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Coupling a Power Dispatch Model with a Wardrop or Mean-Field-Game Equilibrium Model

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Abstract

In this paper, we propose an approach for coupling a power network dispatch model, which is part of a long term multi-energy model, with Wardrop or Mean-Field-Game (MFG) equilibrium models that represent the demand response of a large population of small "prosumers" connected at the various nodes of the electricity network. In a deterministic setting, the problem is akin to an optimization problem with equilibrium constraints taking the form of variational inequalities or nonlinear complementarity conditions. In a stochastic setting, the problem is formulated as a robust optimization with uncertainty sets informed by the probability distributions resulting from an MFG equilibrium solution. Preliminary numerical experimentations, using heuristics mimicking standard price adjustment techniques, are presented for both the deterministic and stochastic cases.

Keywords. Long term energy model, Power generation and distribution submodel, Wardrop equilibrium, Mean Field Game, PHEV strategic charging.

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1 Introduction

An important issue in energy policy is the assessment of the possible contribution of smart grids and demand response to fostering large scale penetration of renewables and thus permitting a transition to low carbon economy. Variable renewable energy (VRE) like, e.g., solar panels and wind mills, are technologies plagued with intermittency, which require secondary reserve that could rapidly compensate a sudden loss of power. These reserves can be provided by battery packs managed by the utility and also by the batteries of a large fleet of EVs and PHEVs[1], provided the owners of these cars (herein called prosumers) participate in a dynamic reserve and demand response (charging schedule) program. Other prosumers, which could provide grid storage are, e.g., the owners of home and office appliances and heating/cooling systems. Prosumers could also be involved in the distributed supply of reactive power compensation[2]. The energy system will thus include a large number of distributed energy resources, managed independently by interconnected agents (the prosumers) that will coordinate through price incentives received via the two-way communication system of a smart-grid (SG). The problem addressed in this paper is the representation of demand response and prosumer behavior effects in global energy models, like ETEM-SG[3], that are currently used to assess the transition to sustainability of energy systems through deep penetration of VRE and SG development.

In [4, 5], it has been shown that a linear programming model, as implemented in ETEM-SG, could take into account this interaction between utility and prosumers, provided one assumes that, in a deterministic environment, the incentives are based on a nodal marginal cost pricing and the prosumers are cost minimizers [6]. In particular, it has been shown in [4] that, through a limit-game argument akin to the Wardrop equilibrium concept [20], one could represent the whole population of prosumers as a single activity within the linear program. Indeed, this implies that all prosumers of a certain category share the same constraints concerning, e.g., the time at which they need to have full usage of their device and, more importantly that they behave as cost minimizers. This is likely an oversimplification and is a motivation for the present paper in which one considers an ETEM-SG model

[1]Electric Vehicles and Plug-in Hybrid Electric Vehicles. In the rest of the paper we shall refer only to PHEVs.

[2]Most of these activities take place at distribution level see [1] and [2] .

[3]ETEM-SG for Energy Technology Environment Model with Smart-Grids is a technology-rich capacity expansion model used to assess long-term energy policies.

where the dispatch submodel is coupled with a Wardrop equilibrium based model that represents the dynamic pricing interaction between the utility and the prosumers. In brief, in the approach proposed in this paper, one represents the interaction between a utility and the many "small" prosumers as an equilibrium in a noncooperative dynamic limit (infinite population) game in both a deterministic and stochastic settings. In the deterministic setting, this approach could be linked with the line of research concerning mathematical programs with complementarity constraints (see [14]), and more particularly multi-level optimization models [10] applied to yield and revenue management [12]. The approach is then extended to a stochastic setting, since demand response is inherently uncertain, due to the random distribution of "state" of the population of prosumers. Since, e.g., a PHEV owner will in general adopt a charging policy in the form of a feedback on the state of charge of the battery and time of the day, the distribution of state of charge of the batteries will result in a stochastic global demand for charging PHEVs.

Recent works have studied prosumer behavior using game theory [33, 34, 19] and in particular, stochastic games, as in [15, 16]. In these works the problem is set at the level of management of a utility which is in relation with small sets of prosumers who play a non-cooperative game in their decision to supply power, storage or reserve. The stochasticity in [33, 34, 19] is related to the intermittency of wind or solar units providing the energy to the prosumers, as well as the fact that individual agent control policies are randomized. Interestingly, the proposed algorithms require very little a priori knowledge, but the price paid is that the training time can become very long if, as in the situation we are studying, the number of agents is very high. The problem considered in this paper is not formulated at the day-to-day management level, but rather at the long term design and planning level. The issue is to find what should be the evolution of the technology mix in a regional power system, in order to reach sustainability, defined, e.g., as a high penetration of renewables in the generation mix. Assessing the contribution of demand response to this endeavor, through the implementation of dynamic time of use pricing is an important issue in these long-term prospective models like TIMES [24] or ETEM-SG [5, 6]. In the present paper we focus on the demand response of PHEV users for charging their vehicles. The number of prosumers considered is not small as we consider a whole population of PHEV users. It is similar to the traffic congestion models where the Wardrop equilibrium concept has emerged [20]. The stochasticity we consider is related to the demand for charging, which will be distributed in the whole population of users. The stochastic game model developed in

[33, 34, 19] would be to no avail for our planning purposes and large number of agents, whereas MFG models will provide an effective tool.

Since we are considering an energy system for a whole region, the number of prosumer agents will be large while each individual agent has very little impact. We could therefore rely on a representation of prosumers behavior as a solution of a stochastic limit game taking the form of an MFG [17]. The question now is *"how can one link an MFG based model of charging PHEV batteries, with a global energy model based on deterministic mathematical programming, in our case, ETEM-SG?"*. The MFG model will typically deliver a probability distribution on the state and actions of a class of prosumers when in a Nash equilibrium within themselves. To integrate this information in a mathematical programming model we propose to use robust optimization (RO) techniques [9, 8, 13, 23, 32] where the design of uncertainty sets is based on the confidence regions obtained from the probability distributions associated with the MFG equilibrium solution. RO methods allow a treatment of uncertainty in large scale models which remains numerically tractable. The coupling will need cobweb style adjustments of price schedules, mimicking the utility virtual reaction to a sequence of announced demand responses. To keep our development centered on the issue of representing demand response effects and for the sake of brevity, we have decided to skip the consideration of uncertainty due to intermittency of VRE technologie, like wind and solar, due to varying weather conditions. In a forthcoming paper we will show how this analysis can be extended to an S-adapted game framework [21] where the players use strategies that are adapted to the unfolding of different possible weather scenarios.

The rest of the paper is organized as follows. In Section 2, we revisit the linkage of ETEM-SG with a (deterministic) Wardrop equilibrium model of prosumers behavior and we propose a procedure for coupling the two models. In Section 3 a Robust Optimization version of ETEM-SG is described. As explained earlier, it is based on first analyzing a collection of MFG based prosumer equilibria at each node of the network. The results of this analysis help characterize a Robust Optimization program at the ETEM level. A heuristic coupling procedure is then proposed to modify the pricing scheme of the prosumers so that their aggregate MFG behavior is consistent with the characteristics of the nodal demands expected by the ETEM-SG program. In both sections a numerical illustration is provided using an ETEM-SG model calibrated on the Leman region in Switzerland as described in [4, 5]. In section 4 we conclude and summarize the contribution of this work, which is essentially methodological.

2 A game theoretic approach to strategic battery charging in a deterministic setting

2.1 ETEM-SG

ETEM-SG[4] is a capacity expansion model used to describe technology choices and investment programs for regional power systems over the long term (time horizon of 50 years or more) under stringent environmental constraints (e.g. satisfying Paris-agreement goals). It is used typically to assess energy transition policies with massive variable renewable energy (VRE) penetration [5] . ETEM-SG includes dispatch and power flow submodels to take into account transmission and distribution constraints and evaluate time of use nodal marginal costs (TUNMC) for a set of typical days. Due to the generalization of smart grid technologies permitting two way communications, the dispatch model could also include, at distribution level, a representation of demand response (e.g., PHEV strategic charging), grid storage and contribution to reactive power compensation. The use of these distributed energy resources (DER) will be driven by market equilibria triggered by dynamic pricing schemes implemented by power utilities. A first attempt to include these features in the model is described in [4, 5] where it is shown that the demand response, e.g. for strategic charging of PHEVs, could be described as the result of a Wardrop equilibrium [20]. Under the assumption that the pricing scheme used by the utility is based on TUNMC and that the users are simply cost minimizers it has also been shown that a global linear program could describe this equilibrium as a DER co-optimization process (see [4]). In the rest of this section we develop in more details the game theoretic approach for representing strategic charging of PHEV in ETEM-SG, and we apply it in a more general setting, which no longer allows a simple solution through linear programming.

2.2 Toward a dynamic locational pricing scheme for battery charging

In smart grid systems, the prosumers behavior will be driven through the dissemination on the transmission and distribution networks of dynamic (time of use) prices. Wholesale electricity pricing in the age of massive renewable penetration is an active area of applied research, see e.g. [25].

[4] Energy-Technology-Environment-Model with Smart-Grid

[5] It is currently used to assess the possible transition to 100% renewable energy in non-interconnected regions (typically islands and remote territories) in France.

Wholesale electricity markets in the United States follow the principles of bid-based, security-constrained, economic dispatch with locational marginal prices. These principles could be adapted to markets for reserve in renewable intensive energy systems [11]. In this subsection we recall the simplest way to obtain locational marginal prices and we propose an "ad hoc" dynamic pricing scheme for PHEV charging, which exploits the locational marginal prices information.

2.2.1 The optimal dispatch subproblem and its dual solution

For a given point in time, the system operator dispatches the committed units so as to minimize the total operational cost. Assume that the generation costs are piecewise linear and denote the vector of nodal generation annualized costs by c_{Gen}.

Assumption 1 *Consider a transmission network with N_b nodes (or buses) linked by L lines described by the following variables and parameters:*

y_n**:** *Net power injection at node $n = 1, \ldots, N_b$; $\mathbf{y}$ is the N_b vector with elements y_n.*

z_ℓ**:** *Flow along line $\ell = 1, \ldots, L$; $\mathbf{z}$ is the L vector with elements z_ℓ.*

$\bar{A}$**:** *Network incidence matrix $L \times N_b$, with $a_{\ell,n} = 1$ if line ℓ originates from n, $a_{\ell,n} = -1$ if line ℓ terminates on n, $a_{\ell,n} = 0$ otherwise. Note that the sum of the columns of A is always equal to the the null column.*

A**:** *An $L \times (N_b - 1)$ matrix obtained by removing a column corresponding to the* swing bus[6] *in the matrix $\bar{A}$.*

$\mathbf{S}$**:** *An $L \times L$ diagonal matrix, $\mathbf{S} = diag(\mathbf{S}_1, \ldots, \mathbf{S}_L)$, where $\mathbf{S}_l$ is the susceptances[7] vector of line l .*

Assume that the demand schedule for PHEV charging is fixed, and defined exogenously.

[6] Usually the swing bus is numbered 1 for the load flow studies. This bus sets the angular reference for all the other buses. Since it is the angle difference between two voltage sources that dictates the real and reactive power flow between them, the particular angle of the swing bus is not important.

[7] In electrical engineering, susceptance ($\mathbf{B}$) is the imaginary part of admittance. The inverse of admittance is impedance and the real part of admittance is conductance. In SI units, susceptance is measured in siemens.

In order to remain within a linear programming framework, it is typical to model power flows using linearized equations[8] can be written as

$$\mathbf{z} = \mathbf{S}A\theta, \tag{1}$$

where θ is the $(N_b - 1)$-vector of angles at the different nodes (buses). Since $\mathbf{y} = A^T\mathbf{z}$, and by introducing A^TSA, one gets:

$$\mathbf{z} = \mathbf{S}A(A^T\mathbf{S}A)^{-1}\mathbf{y}, \tag{2}$$

which can be rewritten

$$\mathbf{z} = \Psi\mathbf{y}, \tag{3}$$

where Ψ is now called the injection shift factor matrix.

The nodal prices are obtained as the dual variables in an optimal dispatch problem under constraints of capacity of the generators and transmission network as shown by Ruiz et al. [28, 29, 30] or Stiel [31].

The distribution of power in the different lines of the transmission network is given by Eq. (2) which we rewrite as follows

$$P_f = \Psi(P_{Gen} - P_{Load}), \tag{4}$$

where $P_f = \mathbf{z}$ is the vector of power flows on each line of the network and $P_{Gen} - P_{Load} = \mathbf{y}$ is the vector of net power injection (generation power P_{Gen} minus load P_{Load}) at each bus (node) of the network.

Notice that at each bus node n, $P_{Load}(n)$ is the sum of conventional (non-flexible) and flexible (in our case PHEV charging denoted $\delta(n,\tau)$ above) loads.

The *transmission sensitivity matrix* $\Psi = \mathbf{S}A(A^T\mathbf{S}A)^{-1}$, also known as the *injection shift factor matrix*, gives the variations in flows due to changes in the nodal injections. The shift factor matrix is a function of the characteristics of the transmission elements and of the state of the transmission switches.

At each time-slice τ, if the demands P_{Load} are exogenously defined the economic dispatch is formulated as a linear program which is summarized as follows:

$$\min_{P_{Gen}} c_{Gen}^T P_{Gen}, \tag{5}$$

[8] This power flow model corresponds to the Approximate DC flow where Power flow obeys Kirchoff'sb voltage law, reactive power is ignored and phase angle differences are small and per unit voltages are set to 1.

under the following set of constraints (with the associated dual variables indicated on the RHS):

$$\mathbf{1}_{N_b}^T(P_{Gen} - P_{Load}) \quad = 0 \qquad (\gamma) \tag{6}$$

$$P_{f\,\min} \leq \Psi\,(P_{Gen} - P_{Load}) \quad \leq P_{f\,\max} \qquad (\mu_{\min}, \mu_{\max}) \tag{7}$$

$$P_{Gen}^{lo} \leq P_{Gen} \quad \leq P_{Gen}^{up}, \qquad (\gamma_{\min}, \gamma_{\max}) \tag{8}$$

where $\mathbf{1}_{N_b}$ stands for an N_b vector whose components are all equal to 1. The constraint (6) ensures the total load-generation balance, (7) enforces the flow limits on transmission elements and flowgates where lower limits usually represent the limit in the opposite flow direction, and (8) models the lower and upper generation limits.

Remark 1 *As is usually the case for dispatch models, we do not consider ramping constraints that would introduce an additional dynamic aspect. These constraints are taken into account in the so called unit commitment models (see e.g. [18]) that operate at a finer and faster time scale.*

It has thus been proved that, under Assumption 1 the following holds true (see [30] for details)

Lemma 1 *The nodal marginal cost vector ϖ is then given by*

$$\varpi = -(\gamma\,\mathbf{1} + \Psi^T(\mu_{\max} - \mu_{\min})). \tag{9}$$

Proof. Apply duality theory.

In addition, we assume that the TUNMCs will serve as a basis for determining an efficient dynamic pricing scheme. However since P_{Load} is partly defined by the behavior of prosumers (the owners of PHEVs deciding independently when to recharge their batteries), there will be a linking to establish between the linear program describing the optimal dispatch, with the computation of marginal nodal prices, and a Wardrop equilibrium model involved in the definition of P_{Load}.

2.3 Wardrop equilibrium induced by nodal marginal prices

To simplify the analysis, we assume that at each node of the transmission grid, also called a bus node, there is a single distribution line to which a set of m_n PHEV batteries are connected.

Assumption 2 *We assume that at each node n, the population of PHEVs is homogenous, composed of m_n identical small users $j = 1, 2, \ldots, m_n$, where m_n is large enough to consider that each prosumer is infinitesimal.*

Remark 2 *Assuming homogeneity of PHEV users may seem very restrictive for a standard dispatch model. However our dispatch model is part of a larger scale energy model. It is standard practice in these global energy models to represent demand devices, like e.g. private cars by one (or a few) typical technology model that is replicated for the whole population of users. For the sake of simplifying the exposition, we have opted for a single class of PHEVs. The approach could easily accommodate several types of electrical vehicles.*

Assume also that the utility will drive the demand for charging the PHEVs batteries by implementing a dynamic pricing scheme where the per car average demand for charging affects the price at different nodes n and time-slices of the day. Such a scheme is for example assumed in the seminal paper [26] proposing decentralized control for the charging of electrical vehicles.

Assumption 3 *The utility will adopt a dynamic pricing scheme for battery charging defined as*

$$p(n, \tau; D(\tau), \Delta(\tau)) = \varpi(n, \tau)(1 + c_1 (\Delta(\tau) - D(\tau))/N), \quad \tau \in \mathcal{T}. \tag{10}$$

Here $\varpi(n, \tau)$ is the nodal marginal cost, for time slice τ, that has been computed from the optimal dispatch model, run with a set of "target" charging demand $d(n, \tau)$ for all nodes n and time slices τ. $\Delta(\tau)$ is the total demand for charging at time-slice τ, $D(\tau)$ is the total "target" charging demand envisioned in the optimal dispatch, at time-slice τ and $N = \sum_n m_n$ is the total number of batteries in the whole population. In (10) the parameter c_1 is calibrating the slope of this inverse demand curve. With this pricing scheme the price increases when real demand exceeds the target and decreases when real demand is lower than the target.

Remark 3 *Notice that this pricing mechanism incorporates both the impact of the transmission network (it is based on nodal marginal cost $\varpi(n, \tau)$) and the discrepancy between the per car average demand for battery charging resulting from prosumers decisions $\Delta(\tau)/N$ and the one that would be preferred by the utility, i.e., $D(\tau)/N$. In our formulation of the linking problem, the target demands should be, at equilibrium, the ones that have served to compute the nodal marginal costs. Therefore, at equilibrium, the prices will be equal to nodal marginal costs, an interesting property for the pricing scheme (10). Notice also that we could have chosen a price mechanism where the per car average demand at each node and not the global per car* *average demand is considered. Such a scheme would reduce the interdependency of the population of prosumers.*

We denote

$\delta(n,\tau)$: The demand for charging PHEVs connected at time-slice τ at the low voltage distribution line linked with bus node n, that will be decided by the prosumers.

$\Delta_{-n}(\tau)$: The vector of expected demands for charging PHEVs connected at time-slice τ at the low voltage distribution lines linked with all other bus nodes (different from n).

Since

$$\Delta(\tau) = \mathbf{1}'\Delta_{-n}(\tau) + \delta(n,\tau),$$

the dynamic pricing scheme (10) is linking together all the PHEV users. To emphasize this point, we shall rewrite the price schedule at node n as $p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))$, i.e. a function of $\delta(n,\tau)$, the total charging demand at node n and $\Delta_{-n}(\tau)$, the vector of all charging demands at other nodes of the network.

At initial time $\tau = 0$ the state of charge of a generic battery at node n is x^o. The evolution of the state of charge of battery j is described by the following state equation

$$x_j(n,\tau+1) = x_j(n,\tau) + \delta_j(n,\tau) - \mathcal{T}_j(n,\tau), \tag{11}$$

where $\delta_j(n,\tau)$ is the charging energy and $\mathcal{T}_j(n,\tau)$ is the exogenously defined energy discharged at time τ due to transport service provided by the PHEV linked to node n. Assume a PHEV user has a utility criterion for the use of the car battery, which it strives to maximize. It is defined as the sum of utilities of charge levels at each time-slice, $u_j(\tau, x_j(n,\tau))$ minus the cost of the charging at each time-slice.

For a given price schedule $p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))$, a PHEV user j, behaving noncooperatively, solves a best response optimal control problem, with control variable δ_j in state equation (11) and criterion

$$\max \sum_{\tau} (u_j(\tau, x_j(n,\tau)) - p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))\delta_j(n,\tau)), \tag{12}$$

with end-point conditions

$$x_j(n,0) = x_j^o \tag{13}$$
$$x_j(n,T) = x_j^o. \tag{14}$$

These boundary conditions state that at the end of each day the state of charge of the battery should be back at the level which is desired at the beginning of the day.

To write the optimality condition, introduce the Hamiltonian

$$\begin{aligned} H_j(x_j(n,\tau),\lambda_j(n,\tau+1),\delta_j(n,\tau)) &= u_j(\tau,x_j(n,\tau)) \\ &\quad -p(n,\tau;d(n,\tau),\Delta_{-n}(\tau))\delta_j(n,\tau) \\ &\quad +\lambda_j(n,\tau+1)\left(\delta_j(n,\tau)-\mathcal{T}_j(n,\tau)\right), \end{aligned}$$

where $\lambda_j(n,\tau+1)$ is the costate or adjoint variable associated with the state equation (11).

The optimality conditions, for $\tau \in \mathcal{T}$ and $n \in \mathcal{N}$, given by the maximum principle of optimal control, are obtained from the derivatives of the Hamiltonian. First we write the complementarity conditions for the decision variable (open-loop strategy) of Player j:

$$\delta_j(n,\tau) \geq 0, \tag{15}$$

$$\begin{aligned} &p'_\delta(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))\delta_j(n,\tau) \\ &\qquad + p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau)) - \lambda_j(n,\tau+1) \geq 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &\delta_j(n,\tau)(p'_\delta(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))\delta_j(n,\tau) \\ &\qquad + p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau)) - \lambda_j(n,\tau+1)) = 0. \end{aligned} \tag{17}$$

Then we write the complementarity conditions for the state variable

$$x_j(n,\tau) \geq 0, \tag{18}$$

$$\lambda_j(n,\tau+1)-\lambda_j(n,\tau)+\nabla_x u_j(\tau,x_j(n,\tau)) \geq 0, \tag{19}$$

$$x_j(n,\tau)(\lambda_j(n,\tau+1)-\lambda_j(n,\tau)+\nabla_x u_j(\tau,x_j(n,\tau))) = 0, \tag{20}$$

$$x_j(n,0)=x_j(n,T) = x^o. \tag{21}$$

Assume that at each node n the population of PHEVs is homogenous, composed of m_n identical small users $j=1,2,\ldots,m_n$, where m_n is large enough to consider that each prosumer is infinitesimal. The total charging demand resulting from the m_n users choosing their charging schedule under a Nash equilibrium condition, is given by

$$\delta(n,\tau) = \sum_{j=1}^{m_n} \delta_j(n,\tau)$$

and the total charges of all batteries connected to node n is given by

$$x(n,\tau) = \sum_{j=1}^{m_n} x_j(n,\tau).$$

Similarly define the global discharge for transport service

$$\mathcal{T}(n,\tau) = \sum_{j=1}^{m_n} \mathcal{T}_j(n,\tau).$$

Then the Nash equilibrium conditions (15–21) can be rewritten at the global level as follows, for $\tau \in \mathcal{T}$ and $n \in \mathcal{N}$,

$$\delta(n,\tau) \geq 0, \tag{22}$$

$$\frac{p'_\delta(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))}{m_n}\delta(n,\tau) \\ + p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau)) - \lambda(n,\tau+1) \geq 0, \tag{23}$$

$$\delta(n,\tau)(\frac{p'_\delta(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))}{m_n}\delta(n,\tau) \\ + p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau)) - \lambda(n,\tau+1)) = 0, \tag{24}$$

and

$$\begin{aligned} x(n,\tau) &\geq 0, & (25)\\ \lambda(n,\tau+1) - \lambda(n,\tau) - m_n \bigtriangledown_x u(\tau, \frac{x(n,\tau)}{m_n}) &\geq 0 & (26)\\ x(n,\tau)(\lambda(n,\tau+1) - \lambda(n,\tau) - m_n \bigtriangledown_x u(\tau, \frac{x(n,\tau)}{m_n})) &= 0 & (27)\\ x(n,0) = x(n,T) &= x^o. & (28) \end{aligned}$$

Definition 2.1 *A Wardrop equilibrium (WE) at node n is the equilibrium in the limit game when* $m_n \to \infty$ *while* $x(n,\tau) = \sum_{j=1}^{m_n} x_j(n,\tau)$ *remains finite, and* $m_n \bigtriangledown_x u(\tau, \frac{x(n,\tau)}{m_n})) \to \bigtriangledown_x \tilde{u}(\tau, x(n,\tau)))$*, where the function* $\tilde{u}(\tau, x(n,\tau)))$ *is the limit of* $m_n u(\tau, \frac{x(n,\tau)}{m_n}))$*.*

So, introducing the costate variable $\lambda(n,\tau+1)$ associated with the state of all the batteries connected to node n we have proved the following lemma which characterizes the WE as the solution of a complementarity problem,

Lemma 2 *In a Wardrop equilibrium the total demand schedule for charging PHEVs $\delta(n, \tau)$, the total state of charge schedule and the costate variables are defined as the solution of the variational inequality (29)-(36), for $\tau \in \mathcal{T}$ and $n \in \mathcal{N}$.*

$$\begin{aligned}
\delta(n,\tau) &\geq 0 && (29)\\
x(n,\tau+1) - x(n,\tau) - \delta(n,\tau) + \mathcal{T}(n,\tau) &= 0 && (30)\\
x(n,\tau+1) &\geq 0 && (31)\\
\lambda(n,\tau+1) - \lambda(n,\tau) - \bigtriangledown_x \tilde{u}(\tau, x(n,\tau)) &\geq 0 && (32)\\
x(n,\tau)(\lambda(n,\tau+1) - \lambda(n,\tau) - \bigtriangledown_x \tilde{u}(\tau, x(n,\tau))) &= 0 && (33)\\
p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau)) + \lambda(n,\tau+1) &\geq 0 && (34)\\
\delta(n,\tau)(p(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau)) + \lambda(n,\tau+1)) &= 0 && (35)\\
x(n,0) = x(n,T) &= x^o. && (36)
\end{aligned}$$

Proof. Pass to the limit in Equations (22)-(28), taking into consideration that the terms

$$\frac{p'_\delta(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))}{m_n}$$

will tend to 0.

Remark 4 *Wardrop equilibria can be viewed as a Nash equilibrium solution of a limit game. It has been shown [20] that they are closely related to competitive market equilibria in economic theory, which are notoriously efficient [3]. In fact it is easy to check that the solution of the Pareto optimality problem where one maximizes the sum of utilities of all players (PHEV users) by selecting optimally the total charging demand would lead a different set of optimality conditions where the term*

$$p'_\delta(n,\tau;\delta(n,\tau),\Delta_{-n}(\tau))\delta(n,\tau)$$

would appear in Eqs (34)-(35). However, in [4] it has been shown that a Wardrop equilibrium describing the strategic charging of PHEVs would lead to an efficient solution if we stay in the realm of linear programming, that is if the payoff of each user is minus the cost and the charging preferences are described by linear constraints on the state of charge.

Remark 5 *Notice that when we get in Subsection 3.2 below to the MFG based analysis, a similar set up of an infinite number of infinitesimal agents* *will be adopted. The main difference later is that the agents cannot be directly lumped into a single macro agent because they will be considered individually stochastic while all agents in the Wardrop framework are identical and deterministic. Thus the MFG framework is less restrictive.*

2.4 Coupling the dispatch model and Wardrop equilibrium at each node n

Let us first define the type of optimality condition we want to look for.

Definition 2.2 *A charging program $\{\delta(n,\tau) : n \in \mathcal{N}, \tau \in \mathcal{T}\}$ is W-optimal if, when introduced in ETEM-SG as target charging demands $(D(\tau) : \tau \in \mathcal{T})$ it yields an optimal dispatch program with marginal nodal prices $\{\varpi(n,\tau) : n \in \mathcal{N}, \tau \in \mathcal{T}\}$, such that the Wardrop equilibrium model with pricing scheme (10), yield again the same charging program $\{\delta(n,\tau) : n \in \mathcal{N}, \tau \in \mathcal{T}\}$.*

According to this definition a W-optimal charging program is a fixed point of the mapping which associates with a target charging demand a new "optimal" charging demand driven by the dynamic pricing scheme and the Wardrop equilibrium behavior of the users. We propose the following scheme to link the optimal dispatch of ETEM-SG with the WE constraints. For all nodes n:

1. **Start:** with a given $\delta^0(n,\tau)$, for example, assuming the utility has a total control on demand. Then at iteration i:

2. **Run ETEM-SG:** with target charging demands $d(n,\tau) = \delta^i(n,\tau)$; From the solution get the vector of nodal marginal costs $\varpi^i(n,\tau)$ for $\tau = 0, \dots, T-1$ and the target charging demands $D(\tau)$;

3. **Run WE model:** The Wardrop equilibrium model is run with a price function $p(n,\tau; D(\tau), \Delta(\tau))$ as defined in (10) and get a new charging demand $\delta^{i+1}(n,\tau)$;

4. **Check convergence:** There is consistency between the two models if $|\delta^i(n,\tau) - \delta^{i+1}(n,\tau)| = 0$. Then **STOP**, otherwise go to step (2) with $i = i + 1$.

Lemma 3 *At convergence of the above linking procedure the Wardrop equilibrium will be defined by the following complementarity conditions, for $\tau \in$*

$\mathcal{T}$ and $n \in \mathcal{N}$,

$$\begin{aligned}
\delta(n,\tau) &\geq 0 && (37)\\
x(n,\tau+1) - x(n,\tau) - \delta(n,\tau) + \mathcal{T}(n,\tau) &= 0 && (38)\\
x(n,\tau+1) &\geq 0 && (39)\\
\lambda(n,\tau+1) - \lambda(n,\tau) - \bigtriangledown_x \tilde{u}(\tau, x(n,\tau)) &\geq 0 && (40)\\
x(n,\tau)(\lambda(n,\tau+1) - \lambda(n,\tau) - \bigtriangledown_x \tilde{u}(\tau, x(n,\tau))) &= 0 && (41)\\
\varpi(n,\tau) + \lambda(n,\tau+1) &\geq 0 && (42)\\
\delta(n,\tau)(\varpi(n,\tau) + \lambda(n,\tau+1)) &= 0 && (43)\\
x(n,0) = x(n,T) &= x^o, && (44)
\end{aligned}$$

where $\varpi(n,\tau)$ is the nodal marginal cost obtained from the dispatch model with charging demands $\delta(n,\tau)$ at each node.

Proof At convergence the target demands and the Wardrop equilibrium demands are the same, hence the price reduces to $\varpi(n,\tau)$ at each node and time slice.

As a corollary to the above result, we see that a W-optimal charging program is given by the solution of a global complementarity problem obtained by appending to the linear complementarity problem associated with the linear program of the dispatch, the nonlinear complementarity problem (38)-(44).

2.5 A numerical illustration

To illustrate this procedure we use an ETEM-SG model calibrated for the Leman region in Switzerland and described in [6]. For the sake of simplifying the experiment, we considered a single bus node and we link the Wardrop equilibrium model with a single typical day of winter 2050 that has 4 time-slices corresponding to (1) morning; (2) afternoon; (3) evening and (4) night. In 2050, the ETEM-SG run estimates that PHEVs will satisfy a third of the mobility demand.

1. **Initial run of ETEM-SG:** In this run the only constraint on charging is to provide the needed energy for satisfying the demand for mobility of PHEVs. The result of the optimal dispatch indicates a charging of 49 TJ during the night time-slice and nodal marginal cost as indicated in Table 1 below.

Table 1: Nodal marginal cost in M$/PJ

Morning	0.32
Afternoon	0.47
Evening	0
Night	0

2. **Run of the Wardrop equilibrium model:** We now run the Wardrop equilibrium model with a price function as defined in (10). We assume that the PHEV owners have a utility function depending on the charge of the battery at each time-slice. More precisely, we assume that they want to track a SOC profile as defined in Table 2 below,

   Table 2: Target SOC ($targ_{soc}(\tau)$ at population level) in PJ

Morning	60
Afternoon	40
Evening	60
Night	10

   with a utility function

   $$\tilde{u}(\tau, x(\tau)) = -0.1(x(\tau) - targ_{soc}(\tau))^2. \quad (45)$$

   The charging demand resulting from the Wardrop equilibrium is obtained, using GAMS with PATH algorithm, as shown in Table 3 below.

   Table 3: Charging demand for Wardrop equilibrium $\delta^*(\tau)$ in PJ

Morning	3.11
Afternoon	7.05
Evening	0
Night	39.44

3. **Check convergence:** We now run ETEM-SG with the additional constraints (where $n \equiv 1$)

$$d(n, \tau) = c_2 \delta^*(n, \tau), \tag{46}$$

with $c_2 = 1$. We observe that the marginal costs do not change. Therefore the WE demand will remain the same and convergence has been reached in two runs of ETEM-SG.

Remark 6 *The very fast convergence of the procedure is due to the fact that, in this example, the flexible load corresponding to electric vehicles is relatively low; therefore this part of the demand has a limited influence on the characterization of nodal marginal costs.*

3 Modeling uncertain charging demand in a Robust ETEM-SG coupled with an FMG

In this section we extend the analysis to a stochastic framework for the description of the charging dynamics for the population of PHEV users. Indeed, the Wardrop based analysis assumes complete solidarity of the states and demands of the vehicles. In reality, even for a homogeneous fleet of vehicles , with identical initial states, the individual energy consumption will differ in general from vehicle to vehicle and will exhibit some randomness. We propose to introduce robust optimization to deal with uncertainty in charging demand and to use an MFG model to represent the probability distribution of charging demands resulting from demand response. In order to simplify the exposition, we restrict the analysis to the case of a single bus node.

3.1 A Robust ETEM-SG formulation

In ETEM-SG, the charging energy delivered at period τ, denoted $\Delta(\tau)$ can be uncertain. This is due to the fact that the population of PHEV users is described by a distribution of the state of charge $x(\tau)$ of the batteries at each time slice τ. The charging decision of a user i is defined by a state feedback law $\theta_i(\tau, \mathbf{x}(\tau))$ and, therefore the demand for battery charging is also random. The total charging energy to be delivered at period τ, denoted $\Delta(\tau) = \sum_{i=1}^{N} \theta_i(\tau, \mathbf{x}(\tau))$, is a random variable.

Consider a probability threshold ε, for example $\varepsilon = 2.5\%$, and introduce a lower bound for the charging demand at time-slice τ based on the lower

ε-quantile $\Delta_\ell(\tau)$, which is defined as follows

$$\Delta_\ell(\tau) = \max\left\{D : \varepsilon \leq \mathrm{P}\left[\Delta(\tau) \leq D\right]\right\}, \quad \tau = 0, \ldots, T-1. \tag{47}$$

Similarly, one can define an upper bound $\Delta_\ell(\tau)$, based on the upper ε-quantile $\Delta_u(\tau)$, which is defined as follows

$$\Delta_u(\tau) = \min\left\{D : \varepsilon \leq \mathrm{P}\left[\Delta(\tau) \geq D\right]\right\}, \quad \tau = 0, \ldots, T-1. \tag{48}$$

For total charging demand, the interval $[\Delta_\ell(\tau), \Delta_u(\tau)]$ has probability $1-2\varepsilon$, i.e. 95% when $\varepsilon = 2.5\%$. We also record the average charging demand, resulting from the MFG equilibrium at each time-slice τ, which is denoted $\Delta_a(\tau)$.

Remark 7 *If the distribution of state of charge is gaussian and the feedback charging control is linear, then the distribution of each random variable $\theta_i(\tau, \mathbf{x}(\tau))$ is also gaussian. In that case the quantiles are proportional to $\sqrt{N}\varsigma(\tau)$ where $\varsigma(\tau)$ is the standard deviation in the distribution of the charging demand for a representative vehicle in the MFG.*

Introduce the random lower and upper bounds that are defined as follows

$$\tilde{\Delta}_\ell(\tau) = \Delta_\ell(\tau) + (\Delta_a(\tau) - \Delta_\ell(\tau))\xi_\ell(\tau) \tag{49}$$

and

$$\tilde{\Delta}_u(\tau) = \Delta_u(\tau) - (\Delta_u(\tau) - \Delta_a(\tau))\xi_u(\tau), \tag{50}$$

where ξ_ℓ and ξ_u are sets of random variables with the following weak assumption.

Assumption 4 *The random factors $\xi_\ell(\tau)$ and $\xi_u(\tau)$ are independent random variables, with common support $[0,1]$ and known means $E(\xi_\ell(\tau)) = \mu_\ell(\tau) \leq \frac{1}{2}$ and $E(\xi_u(\tau)) = \mu_u(\tau) \leq \frac{1}{2}$. Using these definitions, for each time-slice τ the lower bound $\tilde{\Delta}_\ell(\tau)$ ranges in $[\Delta_\ell(\tau), \Delta_a(\tau)]$ and upper bound $\tilde{\Delta}_u(\tau)$ is taking values in $[\Delta_a(\tau), \Delta_u(\tau)]$.*

Now we move to worst case analysis; for that purpose we introduce polyhedral uncertainty sets for ξ_ℓ and ξ_u defined as follows

$$\Xi_\ell = \{\xi_\ell \mid \xi_\ell(\tau) \in [0,1], \forall\tau;\ ||\xi_\ell||_1 \leq k_\ell\}, \tag{51}$$

and

$$\Xi_u = \{\xi_u \mid \xi_u(\tau) \in [0,1], \forall\tau;\ ||\xi_u||_1 \leq k_u\}, \tag{52}$$

where $||\xi_u||_1 = \sum_\tau |\xi_u(\tau)|$, $||\xi_l||_1 = \sum_\tau |\xi_l(\tau)|$ and k_ℓ and k_u are user coefficients limiting the number of random variables taking their worst values simultaneously. Each uncertainty set contains a subset of possible realizations of the uncertainty parameters to be considered in our analysis. Then, one has to look for an optimal solution that remains feasible for all realizations within this uncertainty set, in other words one applies a robust optimization technique [9] on the uncertain lower and upper constraints for the whole set of time-slices $\mathcal{T}$.

$$\sum_{\tau=0}^{T-1} \Delta_\ell(\tau) + (\Delta_a(\tau) - \Delta_\ell(\tau))\xi_\ell(\tau) \leq \sum_{\tau=0}^{T-1} \Delta(\tau), \quad \forall \xi_\ell \in \Xi_\ell \tag{53}$$

and

$$\sum_{\tau=0}^{T-1} \Delta_u(\tau) - (\Delta_u(\tau) - \Delta_a(\tau))\xi_u(\tau) \geq \sum_{\tau=0}^{T-1} \Delta(\tau), \quad \forall \xi_u \in \Xi_u. \tag{54}$$

Based on duality theory, Propositions 1 and 2 show that equations (53) and (54) are equivalent to a set of linear equations.

Proposition 1 *Under Assumption 4, the condition that there exist positive variables $u_\ell \in \mathbf{R}^{|T|}$ and v_ℓ such that the set of deterministic inequalities*

$$\sum_{\tau=0}^{T-1} \Delta_\ell(\tau) + (\Delta_a(\tau) - \Delta_\ell(\tau))\mu_\ell(\tau) + \sum_{\tau=0}^{T-1} u_\ell(\tau)(1 - \mu_\ell(\tau)) + v_\ell \sqrt{\frac{T}{2} ln(\frac{1}{\epsilon})} \leq \sum_{\tau=0}^{T-1} \Delta(\tau), \tag{55}$$

$$u_\ell(\tau) + v_\ell \geq \Delta_a(\tau) - \Delta_\ell(\tau), \quad \tau = 0, \ldots, T-1, \tag{56}$$

is satisfied, guarantees that constraints (53) are satisfied with probability $(1 - \epsilon)$.

Proposition 2 *Under Assumption 4, the condition that there exist positive variables $u_u \in \mathbf{R}^{|T|}$ and v_u such that the set of deterministic inequalities*

$$\sum_{\tau=0}^{T-1} \Delta_u(\tau) - (\Delta_u(\tau) - \Delta_a(\tau))\mu_u(\tau) - \sum_{\tau=0}^{T-1} u_u(\tau)(1 - \mu_u(\tau)) - v_u \sqrt{\frac{T}{2} ln(\frac{1}{\epsilon})} \geq \sum_{\tau=0}^{T-1} \Delta(\tau), \tag{57}$$

$$u_u(\tau) + v_u \geq \Delta_u(\tau) - \Delta_a(\tau), \quad \tau = 0, \ldots, T-1, \tag{58}$$

is satisfied, guaranties that constraints (54) are satisfied with probability $(1 - \epsilon)$.

Propositions 1 and 2 are special cases of a more general theorem from [9] that is proved in [7] and given in Appendix 5.1. These new constraints of Propositions 1 and 2 have to be introduced in ETEM-SG with $\epsilon = 0.05$ in order to insure a satisfaction of 95% for Constraints 53 and 54. We set $\mu_\ell = \mu_u = 1/2$

Remark 8 *This robustification approach has been chosen for its simplicity of implementation. Ultimately though, it would be desirable to approximate chance constraints using appropriate uncertainty sets based, e.g., on CVar [32]. In our approach, lower and upper bounds are uncertain and the model is hedging the risk associated to the worst bound realizations within an uncertainty set. Because variations in (49) and (50) are such that the worst case for each bound individually is at the mean value, the constraints are always feasible. By introducing a robust constraint on the sum of the nodal demands, one takes into account that the worst cases for different time-slices could not occur together with a significant probability. This approach has already been applied in several integrated assessment models [7, 6, 27].*

3.2 An MFG model for battery charging

Let us define precisely the MFG model for battery charging used for determining the distribution of the random charging demand driven by a TOU pricing. Broadly speaking, an MFG is a stochastic game with the following feature; infinitely many indistinguishable players that are interacting through the empirical distribution of the states and/or controls (as it is in our case), of all players. Since the players are indistinguishable, every player can be seen as a generic (or representative) player. Thus the MFG analysis is characterized by the strategic behaviour of a generic agent. By construction, MFG equilibria constitute Nash equilibria for infinite-limit version of large population games with homogeneous players in weak interaction of so-called mean-field type, i.e. depending only on agent state and input empirical distribution. An example of such games is studied here, where the large population of PHEVs interacts dynamically only through the energy price. This interaction is indeed weak (i.e., in an infinite population each PHEV is unaffected by the change in the demand of any other particular PHEV) and of mean-field type (interaction through empirical mean of charging demand only). The applicable aspect of MFG equilibria is their use, albeit under appropriate conditions to be established on a case by case basis, as Approximate Nash Equilibria for any corresponding finite version of the large population games considered. This turns out to be a particularly interesting feature in situations where exact Nash equilibria can not be computed in finite population games. Moreover, crucially, the Approximate Nash equilibria derived from MFG equilibria, exhibits vanishing unilateral deviation gains as the size of the population of players grows to infinity. Precisely, when the Approximate Nash equilibria is implemented, the following is guaranteed: Any player might improve its performance by deviating unilaterally from the equilibria, however the performance gain it can achieve converges to zero as the number of players in the population goes to infinity. Thus, we recover in the infinite-limit version population game (when size of the population is infinite) that the MFG equilibria are indeed Nash equilibria (since any unilateral deviation is faced with a zero performance gain).

Convergence of Nash Equilibria of large population games to corresponding MFG equilibria as the players become infinitesimal is well understood only when the MFG equilibria are unique and the individual feedback best-response function (which is in general a function of the associated Mean Field) is sufficiently smooth. We assume that these conditions are satisfied is this simple instance of MFG solution.

3.2.1 The dynamic pricing scheme

We consider a typical day as time horizon (e.g. the four time slices associated with a typical day $\mathcal{T} = \{\tau_1, \tau_2, \tau_3, \tau_4\}$), a network with a single node (i.e $\mathcal{N} = \{1\}$) with N PHEVs charging their batteries at that single node (i.e $m_1 = N$). We further assume that these N PHEVs are indistinguishable in their strategic charging behaviour. With these assumptions, the price scheme (10), reads as follows : $\forall \tau \in \mathcal{T}$

$$\begin{aligned} p(1,\tau; D(\tau), \Delta(\tau)) \qquad &= \varpi(1,\tau)(1 + c_1 (\Delta(\tau) - D(\tau))/N), \\ &= \varpi(1,\tau)\left(1 + c_1 \left(\frac{\sum_{j=1}^{N} \left[\delta_j(n,\tau) - d_j(n,\tau)\right]}{N}\right)\right). \end{aligned}$$

Under the exchangeability assumption, it makes sense to have the same target demands of all the charging PHEVs resulting from the most current computation of the ETEM-SG program, i.e consider $d(1,\tau)$ such that

$$d_j(1,\tau) = d(1,\tau), \quad \forall \tau \in \mathcal{T}.$$

This has the following consequence : $\forall \tau \in \mathcal{T}$

$$\begin{aligned} p(1,\tau;D(\tau),\Delta(\tau)) \quad &= \varpi(1,\tau)\Bigg(1 + c_1\left(\frac{\sum_{j=1}^N \delta_j(1,\tau) - d_j(1,\tau)}{N}\right)\Bigg), \\ &= \varpi(1,\tau)\Bigg(1 + c_1\left(\frac{\sum_{j=1}^N \delta_j(1,\tau)}{N} - d(1,\tau)\right)\Bigg), \end{aligned}$$

Finally, setting the calibration constant c_1, to be equal to $\frac{1}{d(1,\tau)}$, we obtain,

$$p(1,\tau;D(\tau),\Delta(\tau)) = \frac{\varpi(1,\tau)}{d(1,\tau)}\left(\frac{\sum_{j=1}^N \delta_j(1,\tau)}{N}\right), \quad \forall \tau \in \mathcal{T}. \tag{59}$$

Remark 9 *Observe that the price function (59) obtained, under mild assumptions, from the price function (10) maintains a fundamental feature. That is, the price function is equal to the nodal price, whenever the total realized strategic demand coincides with the total target demand. Indeed,*

$$\forall \tau \in \mathcal{T}, \quad p(1,\tau;D(\tau),\Delta(\tau)) = \varpi(\tau) \Longleftrightarrow \Delta(\tau) = D(\tau).$$

Given this price function, the charging strategies of the batteries are coupled, thus their realized strategic demands form a Nash equilibrium. We denote this equilibrium by $\{\delta_j^*(1,\tau), \forall j = 1,...,N, \forall \tau \in \mathcal{T}\}$*. The associated price at equilibrium is given by*

$$p(1,\tau;D(\tau),\Delta(\tau)) = \frac{\varpi(1,\tau)}{d(1,\tau)}\left(\frac{\sum_{j=1}^N \delta_j^*(1,\tau)}{N}\right), \quad \forall \tau \in \mathcal{T}.$$

This price can be stochastic for any finite N*. However, as* $N \longrightarrow \infty$*, the price becomes deterministic since, by the law of large numbers applied to excheangable demands, it holds that the empirical average of such stochastic demands converges to some deteministic demand, denoted* $\{\bar{\delta}^*(1,\tau), \tau \in \mathcal{T}\}$*. That is,* $\forall \tau \in \mathcal{T}$*,*

$$\frac{\sum_{j=1}^N \delta_j^*(1,\tau)}{N} \longrightarrow \bar{\delta}^*(1,\tau), \quad \text{as} \quad N \longrightarrow \infty.$$

We now describe how this price formulation allows linking the RO disptach model to the MFG model, when $N \longrightarrow \infty$. In the MFG model, at equilibrium, the average demand of the batteries, is deterministic and denoted by $\bar{u} := (\bar{u}_t)_{t\in[0,T]}$. And the instanteneous price function

$$p_t(\bar{u}) := \lambda_1(t)\bar{u}_t + \lambda_0, \quad \forall t \in [0,T],$$

where $\lambda_1 : [0, T] \mapsto [0, +\infty)$, λ_0 is a positive constant. Thus, the RO dispatch model is coherent with the MFG model when :

$$\lambda_0 = 0, \quad \lambda_1(t) := \frac{\bar{w}(1, t)}{d(1, t)} \quad \forall t \in [0, T],$$

where $(\bar{w}(1, t))_{t\in[0,T]}$ and $(d(1, t))_{t\in[0,T]}$, are the continuous versions[9] of nodal price and the target demand at time $t \in [0, T]$.

3.2.2 MFG Formulation

The MFG model for the continuum of batteries charging, is defined in the following two step formulation :

(i) (Strategic Optimization of a generic PHEV/battery) Find a charging policy $u^* \in \mathcal{A}$, such that

$$J(u^*, \bar{u}) = \min_{u \in \mathcal{A}} J(u, \bar{u}), \tag{60}$$

where,

$$\begin{aligned} J(u, \bar{u}) \quad &:= \quad \mathbf{E}\Big[\int_0^T \Big(u_t p_t(\bar{u}) + \frac{q}{2}(X_t - a_t)^2 \Big) dt + \frac{\bar{q}}{2}(X_T - X_0)^2 \Big] \\ &+ \quad \mathbf{E}\Big[\int_0^T \frac{\kappa}{2}(u_t)^2 dt \Big] \end{aligned}$$

with, $X = (X_t)_{t\in[0,T]}$, the evolution for the state of charge (SOC for short) of a generic PHEV/battery. Its dynamics are prescribed by the SDE below : $\forall t \in [0, T]$,

$$X_t = x_0 + \int_0^t (u_r - \nu_r) dr + \epsilon W_t, \quad \mathcal{L}(x_0) = \mathcal{N}(a_0, s^2). \tag{61}$$

The following parameters, assumed to be known and given ; $\nu = (\nu_t)_{t\in[0,T]}$ being the rate of discharge of a generic battery, $a = (a_t)_{t\in[0,T]}$ the SOC profile from which the generic PHEV/battery derives maximum utility, and $\bar{q}, q, \kappa, \epsilon, s$ positive constants used to model the strategic charging behaviour of a generic battery and its initial SOC distribution. The input quadratic regulation term, insures a unique solvability of the best response of the agents, which in turn, allows for

[9] Note that, to obtain time continuous versions of the nodal price and target charging demand at a generic node, we consider spline approximations along the time slices.

the existence of a convergent numerical resolution of the MFG model. Also, the terminal cost parameter $\bar{q}$ must be chosen large enough, so as to enforce the end-point conditions for the batteries SOC, in the RO Dispatch model.

(ii) (Equilibrium Condition) Observe that the optimizer $u^* := (u_t^*)_{t\in[0,T]}$ found above actually depends on the mean field input $\bar{u}$. To assert that $\bar{u}$ is in an MFG equilibrium, one verifies that $\bar{u}$, indeed, coincides with the mean of the realized strategic charging demand of a generic PHEV/battery. That is, we check that $(\bar{u})_{t\in[0,T]}$ satisfies the following fixed point condition ; for all $t \in [0,T]$,

$$\bar{u}_t := \mathbf{E}[u_t^*(\bar{u})]. \tag{62}$$

We say that the MFG model (60-62) has a solution if there is a process, $(u_t^*, \bar{u}_t)_{t\in[0,T]}$ satisfying the fixed-point condition described above. We call this process, an MFG equilibrium.

The MFG defined above is a classical Linear Quadratic MFG, a generalization of this model involving equilibrium through quantiles (and not just the mean) has been studied in [17]. Furthermore, it is well known, that the MFG equilibria are characterized by coupled systems of Hamilton-Jacobi Bellman (HJB for short) and Fokker-Planck (FP for short) Equations. One of the particularities of this system of coupled PDEs is that they are of Forward-Backward nature. Indeed, the HJB equation has a terminal condition while the FP equation has an initial condition. The coupled system of HJB-FP are also simply called the MFG equations.

From the MFG first step, assuming $m(t,x)$ is the density of the generic battery demand distribution, we obtain a value fonction for the control problem, denoted $v(t,x)$, given as the solution to the HJB equation below : $\forall t \in [0,T]$,

$$\begin{aligned}\partial_t v(t,x) \quad &+ \quad \frac{\epsilon^2}{2}\partial_{xx}^2 v(t,x) + \inf_u\{(u-\nu_t)\partial_x v(t,x) + u p_t(\bar{u}) + \frac{\kappa}{2}u^2\} \\ &= \quad -\frac{q(x-a_t)^2}{2}, \quad v(T,x) = 0,\end{aligned}$$

with the optimal demand, given by a feedback policy :

$$\theta(t,x) := -\frac{\partial_x v(t,x) + p_t(\bar{u})}{\kappa}, \quad \forall t \in [0,T].$$

This optimal charging behaviour induces an FP equation for $m(t,x)$, given below : $\forall t \in [0,T]$

$$\begin{aligned}\partial_t m(t,x) &- \frac{\epsilon^2}{2}\partial^2_{xx} m(t,x) - \partial_x\Big[m(t,x)\Big(\frac{p_t(\bar{u}) + \partial_x v(t,x)}{\kappa} + \nu_t\Big)\Big] = 0,\\ m(0,x) &= \frac{1}{\sqrt{2\pi s^2}}\exp\Big(-\frac{(x-a_0)^2}{2s^2}\Big).\end{aligned}$$

Finally, the equilibrium condition reads as follows : $\forall t \in [0,T]$,

$$\bar{u}_t = \int \theta(t,x)m(t,x)dx = -\int \frac{p_t(\bar{u}) + \partial_x v(t,x)}{\kappa} m(t,x)dx. \tag{63}$$

Regrouping these coupled PDEs together, we obtain the following MFG equations for the resolution of our MFG model (60-62) : $\forall t \in (0,T)$

$$\begin{aligned}\partial_t v(t,x) + \frac{\epsilon^2}{2}\partial^2_{xx} v(t,x) + \inf_u\{(u-\nu_t)\partial_x v(t,x) + u p_t(\bar{u}) + \frac{\kappa}{2}u^2\}&\\ + \frac{q(x-a_t)^2}{2} &= 0,\\ \partial_t m(t,x) - \frac{\epsilon^2}{2}\partial^2_{xx} m(t,x) - \partial_x\Big[m(t,x)\Big(\frac{p_t(\bar{u}) + \partial_x v(t,x)}{\kappa} + \nu_t\Big)\Big] &= 0,\\ v(T,x) = 0, \quad m(0,x) = \frac{1}{\sqrt{2\pi s^2}}\exp\Big(-\frac{(x-a_0)^2}{2s^2}\Big) &,\\ \bar{u}_t = -\int \frac{p_t(\bar{u}) + \partial_x v(t,x)}{\kappa} m(t,x)dx, \quad \forall t \in [0,T].&\end{aligned}$$

3.2.3 Solvability of the MFG equations

The following theorem establishes the equivalence between the solvability of our MFG model and the solvability of a system of deterministic Forward Backward Ordinary Differential Equations (FBODEs). The proof of the theorem can be found in [17].

Theorem 1 *There exists an MFG equilibrium $(u^*_t, \bar{u})_{t\in[0,T]}$ to the MFG model (60-62) if and only if there exists a solution $(\eta_t, w_t, \Phi_t, \bar{u})_{t\in[0,T]}$ to* *the system of FBODEs below :*

$$\frac{d\eta_t}{dt} = \frac{\eta_t^2}{\kappa} - q, \quad \eta_T = \bar{q}, \tag{64}$$

$$\frac{d\bar{u}_t}{dt} = -F_1(t)\Phi_t - F_2(t)\bar{u} + F_3(t), \tag{65}$$

$$\frac{d\Phi_t}{dt} = B_1(t)\Phi_t + B_2(t)\bar{u} + B_3(t), \tag{66}$$

$$\bar{u}_0 = I_1(0)\Phi_0 + I_2(0), \quad \Phi_T = -\bar{q}a_T. \tag{67}$$

The time dependent coefficients are defined as ; $\forall t \in [0, T]$*,*

$$\begin{aligned}
I_1(t) &= \frac{-1}{\kappa + \lambda_1(t)}, \quad B_1(t) = \frac{\eta_t}{\kappa}, \quad F_1(t) = -\frac{q}{\eta_t} I_1(t), \\
I_2(t) &= \lambda_0 I_1(t), \quad B_2(t) = \frac{\eta_t \lambda_1(t)}{\kappa}, \quad F_3(t) = I_1(t) q \big[a_t + \frac{\lambda_0}{\eta_t}\big], \\
B_3(t) &= q a_t + \frac{\eta_t(\lambda_0 + \kappa \nu_t)}{\kappa}, \quad F_2(t) = -I_1(t)\left(\frac{d\lambda_1(t)}{dt} + \frac{q}{\eta_t}\big[\lambda_1(t) + \kappa\big]\right).
\end{aligned}$$

Moreover, the feedback control policy defining the MFG equilibrium control, is given by

$$u_t^* = \theta(t, X_t^*) := -\frac{\eta_t X_t^* + \Phi_t + p_t(\bar{u})}{\kappa}, \quad \forall t \in [0, T], \tag{68}$$

where the MFG equilibrium state of charge, is the unique solution to the stochastic differential equation (SDE) bellow : $\forall t \in [0, T]$*,*

$$X_t^* = x_0 + \int_0^t (\theta(t, X_t^*) - \nu_r) dr + \epsilon W_t, \quad \mathcal{L}(x_0) = \mathcal{N}(a_0, s^2). \tag{69}$$

The FBODEs above have been shown to have a unique solution under mild technical assumptions [17]. Thanks to the quadratic regulation, the MFG model is guaranteed to have a unique solution $(u_t^*, \bar{u}_t)_{t \in [0,T]}$, which is given explicitly by Theorem 1. A classical Euler Scheme is used to solve numerically the system of ODEs (64 - 67).

Then, these numerical solutions are averaged over 4 time slices in $\mathcal{T}$, in order to obtain the mean demand of a generic battery at MFG-equilibrium, denoted $\{\bar{u}_\tau, \quad \tau \in \mathcal{T}\}$. Note that one also gets from Theorem 1 the dynamics for the variance of the generic battery's demand at MFG-equilibrium. i.e

$$v_t = \mathbf{V}[u_t^*], \quad \forall t \in [0, T],$$

where $\forall t \in [0, T]$,

$$\frac{dv_t}{dt} = \left(\frac{\epsilon^2 \eta_t^2}{\kappa^2} - \frac{2q}{\eta_t} v_t\right), \quad v_0 = \frac{s^2 \eta_0^2}{\kappa^2}. \tag{70}$$

Since the stochastic stategic demand of a generic battery at MFG-equilibrium can be shown to be normally distributed for all times $t \in [0, T]$ (see 68- 69), prescribing its mean and variance is enough to characterize its distribution. Again, the classical Euler Scheme is used to solve numerically ODE (70) and the solution is averaged over the four time-slices in $\mathcal{T}$, to yield $\{v_\tau, \quad \tau \in \mathcal{T}\}$. For the coupling procedure, given nodal prices from the RO dispatch model, the MFG model is numerically solved and yields the quantities

$$\{(\bar{u}_\tau, v_\tau), \tau \in \mathcal{T}\}, \tag{71}$$

which are in turn, used to build new uncertainty sets for the RO Dispatch model's next run. This coupling procedure is precisely described below.

3.3 A coupling procedure

Definition 3.1 *A charging strategy $u_t^* := \theta^*(\tau, \mathbf{x}^*(\tau))$ is MFG-optimal in ETEM-SG-Robust, if the charging demand confidence intervals defined from (71), the mean and variances of the MFG-equilibrium stochastic demand, as in (49)-(58) yield a robust optimal program with marginal nodal prices $\{\varpi(n, \tau) : n \in \mathcal{N}, \tau \in \mathcal{T}\}$, for which the MFG equilibrium model with pricing scheme (10), produces again the same charging strategy $u_t^* := \theta^*(\tau, \mathbf{x}^*(\tau))$.*

The coupling procedure is summarized as follows:

1. **Start running ETEM-SG** and get a global target charging demand $D(\tau)$ and nodal (in our case a single node) marginal cost $\varpi(1, \tau)$ for $\tau = 0, \dots, T-1$;

2. **Run MFG** with a price function built from $\varpi(1, \tau)$ and $d(1, \tau) = \frac{D(\tau)}{N}$ and get a probability distribution parameters $\{\bar{u}_\tau, v_\tau, \forall \tau \in \mathcal{T}\}$, for the person-by-person optimal charging policy at MFG-equilibrium, denoted $u_\tau^* = \theta^*(\tau, \mathbf{x}^*(\tau)), \forall \tau \in \mathcal{T}$;

3. **Define robust constraints** (53)-(54) and individual bound constraints, and run ETEM-SG with these constraints; if the $\varpi(\tau)$ or $D(\tau)$ do not change **STOP**; otherwise return to Step 2.

By Definition 3.1 we can claim the following

Proposition 3 *Under Assumption 4 with dynamic price scheme (59), if convergence is reached in the procedure described above, then the charging strategy is MFG-optimal in ETEM-SG.*

3.4 A numerical illustration

As for the deterministic case, we illustrate this procedure using the ETEM-SG model calibrated for the Leman region in Switzerland and the MFG algorithm described in [17].

1. **Initial run of ETEM-SG:** As for the deterministic case, the result of the optimal dispatch indicates a charging of 49 TJ during the night time-slice and nodal marginal cost as indicated in Table 1.

2. **Run of the MFG equilibrium model:** We now run the MFG model with a cost function as defined in (61). We assume that the PHEV owners have a utility function depending on the charge of the battery at each time-slice. More precisely, we assume that they want to track a SOC profile as defined in Table 2. The average of the charging demands and its standard deviations resulting from the MFG equilibrium are obtained as shown in Table 4 below.

Table 4: Average and standard deviation of the charging demand from the MFG equilibrium

	Average (Δ_a)	Standard deviation (Δ_σ)
Morning	18.4	8.7
Afternoon	1.9	8.2
Evening	0	7.9
Night	28.3	9.5

3. **Run ETEM-SG robust:** We now run ETEM-SG with the additional robust constraints (53) and (54) where $\Delta_u = \Delta_a + \Delta_\sigma$ and $\Delta_l = \Delta_a - \Delta_\sigma$ and obtain the new charging strategy given in Table 5.

Table 5: Charging demand from ETEM-SG robust

Morning	11.2
Afternoon	0
Evening	0
Night	37.7

We observe that the marginal costs do not change.

4. **Check consistency:** We run the MFG model with the new charging demands computed by ETEM-SG and we observe that the average and standard deviation of the charging demand from the MFG equilibrium remain the same. The two model results are thus consistent and we stop.

4 Discussion and conclusion

We have proposed an approach for considering demand constraints in a power dispatch submodel of a large scale energy model, when these demands are generated by a large number of small prosumers. We have focused on the strategic charging of PHEVs batteries, but the approach could be extended to other types of distributed energy resources provided by prosumers. In a deterministic programming setting, the behavior of prosumers is modeled using the Wardrop equilibrium concept, whereas, in a fully stochastic setting, we propose to use an MFG to compute the non-cooperative equilibrium resulting from the behavior of an ensemble of small independent prosumers.

In fact the Wardrop equilibrium is a simplified MFG equilibrium where agents are considered perfectly deterministic, identical in dynamics, cost and initial conditions. As such, they can be aggregated into a single macro agent representing the total load dynamics. In the MFG case, while the agents are classwise homogeneous, they remain individually stochastic, driven by independent noise processes and starting from independent yet identically distributed random initial conditions. This is a more realistic description and leads to an aggregate behavior which remains stochastic. This stochasticity subsequently informs the formulation of an operator robust optimum power generation scheme.

The physical meaning of the mean-field assumption in an actual grid is that the demand for charging and the supply of distributed energy resources will be inherently stochastic and driven by a strategic response of prosumers to a dynamic pricing scheme. Normally, the per unit cost of power should be a function of total demand and the particular node where the load is being drawn. When the load is stochastic, as for example in the case of a collection of PHEVs, this would result in a highly unpredictable and fluctuating pricing scheme. This is an undesirable situation for both the operator and the individual agent. By assuming that the number of connected cars is roughly stable, the mean load per car becomes a reasonable measure, albeit a risk neutral one, of the global PHEVs consumption. Thus the operator could announce that it will be using the trajectory of mean demand per car to help set the unit price of power. Interestingly, cars are individually optimizing agents in a game where the only coupling of their otherwise independent battery dynamics occurs through cost. As the number of PHEVs increases indefinitely, their pairwise coupling gets weaker and weaker, and eventually, the law of large numbers kicks in to make the mean demand trajectory deterministic. This is the mean field limit. It is a useful state because at that point individual cars have to solve an optimal control problem, and no longer a much more complex game, since the mass behavior becomes insensitive to individual actions. The mean field limit demand trajectory is then obtained by requiring that it is replicated as the mean demand of individuals under their best response to the associated pricing trajectory (basically a fixed point calculation). The result is a predictable pricing trajectory (a desirable result) which is a Nash equilibrium for the infinite population, and an approximate Nash for the actual finite population with possible gains of a single agent deviating from the control policy associated with the mean field equilibrium provably going to zero as the number of agents goes to infinity [20]. Still, the cars global demand remains stochastic and if one characterizes the evolution of its probability distribution, the operator can use that information to develop computationally tractable optimal power generation schemes with adequate robustness properties.

By introducing a linking procedure, based on the use of a dynamic pricing scheme, we are able to circumvent the difficulty arising from the introduction of nonlinear complementarity constraints (in the deterministic case) or complex chance constraints in the fully stochastic case. The price adjustment is indeed ad hoc, indeed, although although it exhibits as we shall argue interesting properties. Note that it includes the nodal marginal cost (there is a general trend currently to extend marginal cost pricing to nodal and time of use marginal cost pricing (see [22])). In practice the precise marginal cost computation may be quite difficult. If we had chosen to include dynamic nodal marginal cost in an equilibrium model we will end up with a hierarchical two-level optimization problem, too complicated. Our pricing scheme and the two-model dialogue is, instead, a heuristic. It permits the inclusion of PHEVs strategic charging in the dispatch submodel of a global capacity expansion model used to assess transition to sustainability. The proposed price adjustment scheme has the interesting property that it modifies the "official" marginal costs in such a way that mean car demand ends up following a trajectory such that on that trajectory, marginal costs initially calculated for an ideal demand scenario apply. So, ultimately, cars end up paying for their realized demand at the "ideal" marginal costs. While they were forbidden from deviating from the ideal demand through adequately modified marginal costs, *no trace* of this modification persists on the final car demand trajectory.

Simple numerical experiments showed how the procedure works in both deterministic and stochastic settings. Future research work includes extensive applications of the proposed linking procedure. In particular this game theoretic structure used to represent the strategic charging of PHEVs should be extended to other types of interactions between utilities and prosumers; we refer in particular to new markets for distributed energy resources, like e.g., secondary reserve to mitigate intermittency of wind and solar power, reactive power compensation, and other system services required to foster massive renewables penetration in power systems.

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5 Appendix

5.1 Proof of Propositions 1 and 2

Propositions 1 and 2 are special cases of a more general theorem in RO, by taking into account that $\Delta_a(\tau) \leq \Delta_\ell(\tau)$ and $\Delta_u(\tau) \leq \Delta_a(\tau)$, respectively. For the interested readers, we state now the general theorem and prove Proposition 1 as a corollary. In order to show the derivation we shall use the concise notation

$$z_0 + \sum_\tau z_\tau \xi_\tau \leq 0 \tag{72}$$

to represent the inequality (53). The coefficients $\hat{z}$ are easily identified as

$$\begin{aligned} z_0 &= \sum_{\tau=0}^{T-1} \Delta_\ell(\tau) \\ z_\tau &= \Delta_a(\tau) - \Delta_\ell(\tau). \end{aligned}$$

The proof is similar for Proposition 2 considering

$$\begin{aligned} z_0 &= \sum_{\tau=0}^{T-1} \Delta_u(\tau) \\ z_\tau &= -(\Delta_u(\tau) - \Delta_a(\tau)). \end{aligned}$$

The main result can be formulated as

Theorem 2 *Let η_τ be T independent random variables with common support $[-1,1]$ and known means $E(\eta_\tau) = \nu_\tau$. The probabilistic inequality $\hat{z}_0 + \sum_\tau \hat{z}_\tau \eta_\tau \leq 0$ is satisfied with probability at least $(1-\epsilon)$ if there exists a vector $w \in \mathbb{R}^\tau$ such that the deterministic inequality*

$$\hat{z}_0 + \sum_\tau \hat{z}_\tau \nu_\tau + \sum_\tau (|w_k| - w_\tau \nu_\tau) + \sqrt{2T \ln \frac{1}{\epsilon}} \max_\tau |\hat{z}_\tau - w_\tau| \leq 0 \qquad (73)$$

is satisfied.

Note that the range of the random factors is now $[-1,1]$. The above theorem is the formal statement of the theory for inequalities with random factors having known means ν_τ and common range $[-1,1]$ as discussed in [9, example 2.4.9, p. 55].

We show now how to prove Proposition 1 as a corollary of Theorem 2.

Proof 1 (Proposition 1) *Let us start with (72) and define the variables $\eta_\tau = 2\xi_\tau - 1$. In view of Assumption 4 the range of η_τ is $[-1,1]$ and $E(\eta_\tau) = \nu_\tau = 2\mu_\tau - 1 \leq 0$. Inequality (72) becomes*

$$z_0 + \frac{1}{2}\sum_\tau z_\tau + \frac{1}{2}\sum_\tau z_\tau \eta_\tau \leq 0.$$

Let $\hat{z}_0 = z_0 + \frac{1}{2}\sum_\tau z_\tau$ and $\hat{z}_\tau = z_\tau/2$. The hypotheses of Theorem 2 for the inequality $\hat{z}_0 + \sum_k \hat{z}_k \eta_k \leq 0$ are verified. Hence,

$$\hat{z}_0 + \sum_\tau \hat{z}_\tau \nu_\tau + \sum_\tau (|w_\tau| - w_\tau \nu_\tau) + \sqrt{2T \ln \frac{1}{\epsilon}} \max_\tau |\hat{z}_\tau - w_\tau| \leq 0$$

is a sufficient condition to ensure the constraint satisfaction with probability at least $(1-\epsilon)$. *If we substitute* ν_τ, $\hat{z}_0$ *and* $\hat{z}_\tau$ *by their values, we obtain the condition*

$$z_0 + \sum_\tau \mu_\tau z_\tau + \sum_\tau (|w_\tau| + w_\tau - 2\mu_\tau w_\tau) + \sqrt{\frac{T}{2}\ln\frac{1}{\epsilon}}\max_\tau |z_\tau - 2w_\tau| \leq 0. \quad (74)$$

Recall that $z_\tau = \Delta_a(\tau) - \Delta_\ell(\tau) \geq 0$. *We claim that only positive values* $w \geq 0$ *need to be considered. Indeed, the theorem does not specify the value it should take. In particular, we can choose* w *so as to have to minimize the right-most component* $\sum_\tau (|w_\tau| + w_\tau - 2\mu_\tau w_\tau) + \sqrt{\frac{T}{2}\ln\frac{1}{\epsilon}}\max_k |z_\tau - 2w_\tau|$. *If for some* τ', $w_{\tau'} < 0$, *then* $|w_{\tau'}| + w_{\tau'} = 0$ *and the contribution of term* τ' *in the summation is* $-2\mu_{\tau'}w_{\tau'} + \max\{z_{\tau'} - 2w_{\tau'}, \max_{\tau\neq\tau'}|z_\tau - w_\tau|\}$. *Clearly this term can be made smaller by taking* $w_{\tau'} = 0$. *Hence, we can assume* $w_{\tau'} \geq 0$.

With $w \geq 0$ *inequality (74) becomes*

$$z_0 + \sum_\tau \mu_\tau z_\tau + \sum_\tau (1-\mu_\tau)2w_\tau + \sqrt{\frac{T}{2}\ln\frac{1}{\epsilon}}\max_\tau |z_\tau - 2w_\tau| \leq 0.$$

Writing u *for* $2w$ *in the above inequality, we obtain the condition*

$$z_0 + \sum_\tau \mu_\tau z_\tau + \sum_\tau (1-\mu_\tau)u_\tau + \sqrt{\frac{T}{2}\ln\frac{1}{\epsilon}}\max_\tau |z_\tau - u_\tau| \leq 0.$$

Using the same argument as before, we easily prove that we can restrict our choice of u *to* $u \leq z$. *Hence,* $|z_\tau - u_\tau| = z_\tau - u_\tau \geq 0$ *and using the additional scalar variable* $v \geq 0$ *we can transform our inequality into*

$$\begin{aligned} z_0 + \sum_\tau \mu_\tau z_\tau + \sum_\tau (1-\mu_\tau)u_\tau + \sqrt{\frac{T}{2}\ln\frac{1}{\epsilon}}\cdot v &\leq 0 \\ z_\tau - u_\tau &\leq v,\ \tau = 0,\ldots,T-1 \\ u \geq 0, v \geq 0. \end{aligned}$$

This concludes the proof of the proposition. ∎

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